From the observations of Barkla and Simons and Barkla and Philpot, it is shown that the coefficient of absorption of a beam of X-radiation in a gas is proportional to the quantity of electronic radiation liberated from the gas, and hence the X-radiation absorbed by an atom of a given element will also be proportional to the fourth power of the weight of the absorbing atom. Recent experiments of Bragg and Pierce show that a similar law holds good for elements in the solid state, and is therefore probably of universal application: it should therefore be possible to calculate the absorption coefficient of any material, provided its homogeneous radiation is not excited.

These points are under further investigation.

I should like to express my indebtedness to Prof. O. W. Richardson for the interest he has taken in these investigations, and for the facilities he has provided for their furtherance.

Deep Water Waves, Progressive or Stationary, to the Third Order of Approximation.

By Lord Rayleigh, O.M., F.R.S.

(Received March 26, 1915.)

As is well known, the form of periodic waves progressing over deep water *without change of type* was determined by Stokes* to a high degree of approximation. The wave-length (λ) in the direction of x being 2π and the velocity of propagation unity, the form of the surface is given by

$$y = a\cos(x-t) - \tfrac{1}{2}a^2\cos 2(x-t) + \tfrac{3}{8}a^3\cos 3(x-t), \tag{1}$$

and the corresponding gravity necessary to maintain the motion by

$$g = 1 - a^2. \tag{2}$$

The generalisation to other wave-lengths and velocities follows by "dimensions."

These and further results for progressive waves of permanent type are most easily arrived at by use of the stream-function on the supposition that the waves are reduced to rest by an opposite motion of the water as a whole, when the problem becomes one of steady motion.† My object at present is to extend the scope of the investigation by abandoning the initial restriction to progressive waves of permanent type. The more general equations may then be applied to progressive waves as a particular case, or to stationary waves in which the principal motion is proportional to a simple circular function of the time, and further to ascertain what occurs when the conditions necessary for the particular cases are not satisfied. Under these circumstances the use of the stream-function loses much of its advantage, and the method followed is akin to that originally adopted by Stokes.

---

* 'Camb. Phil. Trans.,' vol. 8, p. 441 (1847); 'Math. and Phys. Papers,' vol. 1, p. 197.

† 'Phil. Mag.,' vol. 1, p. 257 (1876); 'Scientific Papers,' vol. 1, p. 262. Also 'Phil. Mag.,' vol. 21, p. 183 (1911).

The velocity-potential ϕ, being periodic in x, may be expressed by the series

$$\phi = \alpha e^{-y}\sin x - \alpha' e^{-y}\cos x + \beta e^{-2y}\sin 2x - \beta' e^{-2y}\cos 2x + \gamma e^{-3y}\sin 3x - \gamma' e^{-3y}\cos 3x + \ldots, \qquad (3)$$

where α, α', β, etc., are functions of the time only, and y is measured downwards from mean level. In accordance with (3) the component velocities are given by

$$u = d\phi/dx = e^{-y}(\alpha\cos x + \alpha'\sin x) + 2e^{-2y}(\beta\cos 2x + \beta'\sin 2x) + \ldots$$

$$-v = d\phi/dy = e^{-y}(\alpha\sin x - \alpha'\cos x) + 2e^{-2y}(\beta\sin 2x - \beta'\cos 2x) + \ldots$$

The density being taken as unity, the pressure equation is

$$p = -d\phi/dt + \mathrm{F} + gy - \tfrac{1}{2}(u^2 + v^2), \qquad (4)$$

in which F is a function of the time.

In applying (4) we will regard α, α', as small quantities of the first order, while β, β', γ, γ', are small quantities of the second order at most; and for the present we retain only quantities of the second order. β, etc., will then not appear in the expression for $u^2 + v^2$. In fact

$$u^2 + v^2 = e^{-2y}(\alpha^2 + \alpha'^2),$$

and

$$p = -\frac{d\alpha}{dt}e^{-y}\sin x + \frac{d\alpha'}{dt}e^{-y}\cos x - \frac{d\beta}{dt}e^{-2y}\sin 2x + \ldots + gy - \tfrac{1}{2}e^{-2y}(\alpha^2 + \alpha'^2) + \mathrm{F}. \qquad (5)$$

The surface conditions are (i) that p be there zero, and (ii) that

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} = 0. \qquad (6)$$

The first is already virtually expressed in (5). For the second

$$\frac{dp}{dt} = -\frac{d^2\alpha}{dt^2}e^{-y}\sin x + \frac{d^2\alpha'}{dt^2}e^{-y}\cos x - \ldots - e^{-2y}\left(\alpha\frac{d\alpha}{dt} + \alpha'\frac{d\alpha'}{dt}\right) + \mathrm{F}',$$

$$\frac{dp}{dx} = -\frac{d\alpha}{dt}e^{-y}\cos x - \frac{d\alpha'}{dt}e^{-y}\sin x - \ldots,$$

$$\frac{dp}{dy} = \frac{d\alpha}{dt}e^{-y}\sin x - \frac{d\alpha'}{dt}e^{-y}\cos x + \ldots + g + e^{-2y}(\alpha^2 + \alpha'^2).$$

In forming equation (6) to the second order of small quantities we need to include only the principal term of u, but v must be taken correct to the second order. As the equation of the free surface we assume

$$y = a\cos x + a'\sin x + b\cos 2x + b'\sin 2x + c\cos 3x + c'\sin 3x + \ldots \tag{7}$$

in which b, b', c, c', are small compared with a, a'. Thus (6) gives

$$\begin{aligned}(1-a\cos x-a'\sin x)&\left(-\frac{d^2\alpha}{dt^2}\sin x+\frac{d^2\alpha'}{dt^2}\cos x\right)-\frac{d^2\beta}{dt^2}\sin 2x\\ &+\frac{d^2\beta'}{dt^2}\cos 2x-\frac{d^2\gamma}{dt^2}\sin 3x+\frac{d^2\gamma}{dt^2}\cos 3x-\alpha\frac{d\alpha}{dt}-\alpha'\frac{d\alpha'}{dt}+\mathrm{F}'\\ &-(\alpha\cos x+\alpha'\sin x)\left(\frac{d\alpha}{dt}\cos x+\frac{d\alpha'}{dt}\sin x\right)-\{(1-a\cos x-a'\sin x)\\ &\times(\alpha\sin x-\alpha'\cos x)+2\beta\sin 2x-2\beta'\cos 2x+3\gamma\sin 3x-3\gamma'\cos 3x\}\\ &\times\left\{g+\frac{d\alpha}{dt}\sin x-\frac{d\alpha'}{dt}\cos x\right\}=0.\end{aligned} \tag{8}$$

This equation is to hold good to the second order for all values of x, and therefore for each Fourier component separately. The terms in $\sin x$ and $\cos x$ give

$$\frac{d^2\alpha}{dt^2}+g\alpha = 0, \qquad \frac{d^2\alpha'}{dt^2}+g\alpha' = 0. \tag{9}$$

The term in $\sin 2x$ gives

$$\frac{d^2\beta}{dt^2}+2g\beta = \frac{a}{2}\left(\frac{d^2\alpha}{dt^2}+g\alpha\right)-\frac{a'}{2}\left(\frac{d^2\alpha'}{dt^2}+g\alpha'\right) = 0, \tag{10}$$

and, similarly, that in $\cos 2x$ gives

$$\frac{d^2\beta'}{dt^2}+2g\beta' = 0. \tag{11}$$

In like manner

$$\frac{d^2\gamma}{dt^2}+3g\gamma = 0, \qquad \frac{d^2\gamma'}{dt^2}+3g\gamma' = 0, \tag{12}$$

and so on. These are the results of the surface condition $\mathrm{D}p/\mathrm{D}t = 0$. From the other surface condition ($p = 0$) we find in the same way

$$-\frac{d\alpha}{dt}+ga' = 0, \qquad \frac{d\alpha'}{dt}+ga = 0. \tag{13}$$

$$gb' = \frac{d\beta}{dt}+\frac{a'}{2}\frac{d\alpha'}{dt}-\frac{a}{2}\frac{d\alpha}{dt} = \frac{d\beta}{dt}-gaa'. \tag{14}$$

$$gb = -\frac{d\beta}{dt}+\frac{a'}{2}\frac{d\alpha}{dt}+\frac{a}{2}\frac{d\alpha'}{dt} = -\frac{d\beta}{dt}+\tfrac{1}{2}g(a'^2-a^2). \tag{15}$$

$$-\frac{d\gamma}{dt}+gc' = 0, \qquad \frac{d\gamma'}{dt}+gc = 0. \tag{16}$$

From equations (9) to (16) we see that a, a', satisfy the same equations (9) as do α, α', and also that c, c', satisfy the same equations (12) as do γ, γ'; but that b, b', are not quite so simply related to β, β'.

Let us now suppose that the principal terms represent a progressive wave. In accordance with (9) we may take

$$a = \mathrm{A}\cos t', \qquad a' = \mathrm{A}\sin t', \tag{17}$$

where $t' = \sqrt{g}\,.\,t$. Then if β, β', γ, γ', do not appear, c, c', are zero, and $b = \frac{1}{2}\mathrm{A}^2(\sin^2 t' - \cos^2 t')$, $b' = -\mathrm{A}^2\cos t' \sin t'$; so that

$$y = \mathrm{A}\cos(x-t') - \tfrac{1}{2}\mathrm{A}^2\cos 2(x-t'), \tag{18}$$

representing a permanent wave-form propagated with velocity $\sqrt{g}$. So far as it goes, this agrees with (1). But now in addition to these terms we may have others, for which b, b' need only to satisfy

$$(d^2/dt'^2+2)(b, b') = 0, \tag{19}$$

and c, c', need only to satisfy

$$(d^2/dt'^2+3)(c, c') = 0. \tag{20}$$

The corresponding terms in y represent merely such waves, propagated in either direction, and of wave-lengths equal to an aliquot part of the principal wave-length, as might exist alone of infinitesimal height, when there is no primary wave at all. When these are included, the aggregate, even though it be all propagated in the same direction, loses its character of possessing a permanent wave shape, and further it has no tendency to acquire such a character as time advances.

If the principal wave is *stationary* we may take

$$a = \mathrm{A}\cos t', \qquad a' = 0. \tag{21}$$

If β, β', γ, γ', vanish,

$$b = -\tfrac{1}{2}a^2, \quad b' = 0, \quad c = 0, \quad c' = 0,$$

and

$$y = \mathrm{A}\cos x\,.\,\cos t' - \tfrac{1}{2}\mathrm{A}^2\cos 2x\,.\,\cos^2 t'. \tag{22}$$

According to (22) the surface comes to its zero position everywhere when $\cos t' = 0$, and the displacement is a maximum when $\cos t' = \pm 1$. Then

$$y = \pm\mathrm{A}\cos x - \tfrac{1}{2}\mathrm{A}^2\cos 2x, \tag{23}$$

so that at this moment the wave-form is the same as for the progressive wave (18). Since y is measured downwards, the maximum elevation above the mean level exceeds numerically the maximum depression below it.

In the more general case (still with β, etc., evanescent) we may write

$$a = \mathrm{A}\cos t' + \mathrm{B}\sin t', \quad a' = \mathrm{A}'\cos t' + \mathrm{B}'\sin t',$$

with

$$b' = -aa', \quad b = \tfrac{1}{2}(a'^2 - a^2), \quad c' = 0, \quad c = 0.$$

When β, β', γ, γ', are finite, waves such as might exist alone, of lengths equal to aliquot parts of the principal wave-length and of corresponding frequencies, are superposed. In these waves the amplitude and phase are arbitrary.

When we retain the third order of small quantities, the equations naturally become more complicated. We now assume that in (3) β, β', are small quantities of the second order, and γ, γ', small quantities of the third order. For p, as an extension of (5), we get

$$\begin{aligned} p = {} & e^{-y}\left(-\frac{d\alpha}{dt}\sin x+\frac{d\alpha'}{dt}\cos x\right)+e^{-2y}\left(-\frac{d\beta}{dt}\sin 2x+\frac{d\beta'}{dt}\cos 2x\right) \\ & +e^{-3y}\left(-\frac{d\gamma}{dt}\sin 3x+\frac{d\gamma'}{dt}\cos 3x\right)+gy+\mathrm{F}-\tfrac{1}{2}e^{-2y}(\alpha^2+\alpha'^2) \\ & -2e^{-3y}\{(\alpha\beta+\alpha'\beta')\cos x+(\alpha\beta'-\alpha'\beta)\sin x\}. \end{aligned} \tag{24}$$

This is to be made to vanish at the surface. Also we find, on reduction,

$$\begin{aligned} -\frac{\mathrm{D}p}{\mathrm{D}t} = {} & (1-y+\tfrac{1}{2}y^2)\left\{\left(\frac{d^2\alpha}{dt^2}+g\alpha\right)\sin x-\left(\frac{d^2\alpha'}{dt^2}+g\alpha'\right)\cos x\right\} \\ & +(1-2y)\left\{\left(\frac{d^2\beta}{dt^2}+2g\beta\right)\sin 2x-\left(\frac{d^2\beta'}{dt^2}+2g\beta'\right)\cos 2x\right\} \\ & +\left(\frac{d^2\gamma}{dt^2}+3g\gamma\right)\sin 3x-\left(\frac{d^2\gamma'}{dt^2}+3g\gamma'\right)\cos 3x-\mathrm{F}' \\ & +2(1-2y)\left(\alpha\frac{d\alpha}{dt}+\alpha'\frac{d\alpha'}{dt}\right)+4\sin x\frac{d}{dt}(\alpha\beta'-\alpha'\beta) \\ & +4\cos x\frac{d}{dt}(\alpha\beta'+\alpha'\beta)+(\alpha^2+\alpha'^2)(\alpha\sin x-\alpha'\cos x)\,; \end{aligned} \tag{25}$$

and at the surface $\mathrm{D}p/\mathrm{D}t = 0$ for all values of x. In (25) y is of the form (7), where b, b', are of the second order, c, c', of the third order.

Considering the coefficients of $\sin x$, $\cos x$, in (25) when reduced to Fourier's form, we see that $d^2\alpha/dt^2+g\alpha$, $d^2\alpha'/dt^2+g\alpha'$, are both of the third order of small quantities, so that in the first line the factor $(1-y+\frac{1}{2}y^2)$ may be replaced by unity. Again, from the coefficients of $\sin 2x$, $\cos 2x$, we see that to the third order inclusive

$$\frac{d^2\beta}{dt^2}+2g\beta = 0, \qquad \frac{d^2\beta'}{dt^2}+2g\beta' = 0, \tag{26}$$

and from the coefficients of $\sin 3x$, $\cos 3x$, that to the third order inclusive

$$\frac{d^2\gamma}{dt^2}+3g\gamma = 0, \qquad \frac{d^2\gamma'}{dt^2}+3g\gamma' = 0. \tag{27}$$

And now returning to the coefficients of sin x, cos x, we get

$$\frac{d^2\alpha}{dt^2}+g\alpha-2a'\frac{d}{dt}(\alpha^2+\alpha'^2)+4\frac{d}{dt}(\alpha\beta'-\alpha'\beta)+\alpha(\alpha^2+\alpha'^2)=0, \tag{28}$$

$$\frac{d^2\alpha'}{dt^2}+g\alpha'+2a\frac{d}{dt}(\alpha^2+\alpha'^2)-4\frac{d}{dt}(\alpha\beta'+\alpha'\beta)+\alpha'(\alpha^2+\alpha'^2)=0. \tag{29}$$

Passing next to the condition $p=0$, we see from (24), by considering the coefficients of sin x, cos x, that

$$-\frac{d\alpha}{dt}+ga'+\text{terms of 3rd order}=0,$$

$$\frac{d\alpha'}{dt}+ga+\text{terms of 3rd order}=0.$$

The coefficients of sin $2x$, cos $2x$, require, as in (14), (15), that

$$b'=\frac{1}{g}\frac{d\beta}{dt}-aa', \qquad b=-\frac{1}{g}\frac{d\beta'}{dt}+\frac{a'^2-a^2}{2}. \tag{30}$$

Again, the coefficients of sin $3x$, cos $3x$, give

$$\begin{aligned} c' &= \frac{1}{g}\frac{d\gamma}{dt}-\tfrac{3}{2}(a'b+ab')+\tfrac{3}{8}a'(a'^2-3a^2) \\ &= \frac{1}{g}\left\{\frac{d\gamma}{dt}-\frac{3a}{2}\frac{d\beta}{dt}+\frac{3a'}{2}\frac{d\beta'}{dt}\right\}-\frac{3a'}{8}(a'^2-3a^2), \end{aligned} \tag{31}$$

$$\begin{aligned} c &= -\frac{1}{g}\frac{d\gamma'}{dt}+\tfrac{3}{2}(a'b'-ab)+\frac{3a}{8}(3a'^2-a^2) \\ &= \frac{1}{g}\left\{-\frac{d\gamma'}{dt}+\frac{3a'}{2}\frac{d\beta}{dt}+\frac{3a}{2}\frac{d\beta'}{dt}\right\}-\frac{3a}{8}(3a'^2-a^2). \end{aligned} \tag{32}$$

When β, β', γ, γ', vanish, these results are much simplified. We have

$$b'=-aa', \qquad b=\tfrac{1}{2}(a'^2-a^2), \tag{33}$$

$$c'=-\frac{3a'}{8}(a'^2-3a^2), \qquad c=-\frac{3a}{8}(3a'^2-a^2). \tag{34}$$

If the principal terms represent a purely progressive wave, we may take, as in (17),

$$a=\mathrm{A}\cos nt, \qquad a'=\mathrm{A}\sin nt, \tag{35}$$

where n is for the moment undetermined. Accordingly

$$b'=-\tfrac{1}{2}\mathrm{A}^2\sin 2nt, \qquad b=-\tfrac{1}{2}\mathrm{A}^2\cos 2nt,$$

$$c'=\tfrac{3}{8}\mathrm{A}^3\sin 3nt, \qquad c=\tfrac{3}{8}\mathrm{A}^3\cos 3nt;$$

so that

$$y=\mathrm{A}\cos(x-nt)-\tfrac{1}{2}\mathrm{A}^2\cos 2(x-nt)+\tfrac{3}{8}\mathrm{A}^3\cos 3(x-nt), \tag{36}$$

representing a progressive wave of permanent type, as found by Stokes.

To determine n we utilise (28), (29), in the small terms of which we may take

$$\alpha = g\int a'dt = -\frac{gA}{n}\cos nt, \qquad \alpha' = -g\int a\,dt = -\frac{gA}{n}\sin nt,$$

so that

$$\alpha^2+\alpha'^2 = g^2A^2/n^2.$$

Thus

$$\frac{d^2(\alpha,\,\alpha')}{dt^2}+\left(g+\frac{g^2A^2}{n^2}\right)(\alpha,\,\alpha') = 0,$$

and

$$n^2 = g+g^2A^2/n^2 = g(1+A^2), \tag{37}$$

or, if we restore homogeneity by introduction of $k(=2\pi/\lambda)$,

$$n^2 = g/k\,.\,(1+k^2A^2). \tag{38}$$

Let us next suppose that the principal terms represent a stationary, instead of a progressive, wave and take

$$a = A\cos nt, \qquad a' = 0. \tag{39}$$

Then by (33), (34),

$$b' = 0, \qquad b = -\tfrac{1}{2}A^2\cos^2 nt, \qquad c' = 0, \qquad c = \tfrac{3}{8}A^3\cos^3 nt;$$

and

$$y = A\cos nt\cos x-\tfrac{1}{2}A^2\cos^2 nt\cos 2x+\tfrac{3}{8}A^3\cos^3 nt\cos 3x. \tag{40}$$

When $\cos nt = 0$, $y = 0$ throughout; when $\cos nt = 1$,

$$y = A\cos x-\tfrac{1}{2}A^2\cos 2x+\tfrac{3}{8}A^3\cos 3x,$$

so that at this moment of maximum displacement the form is the same as for the progressive wave (36).

We have still to determine n so as to satisfy (28), (29), with evanescent β, β'. The first is satisfied by $\alpha = 0$, since $a' = 0$. The second becomes

$$\frac{d^2\alpha'}{dt^2}+g\alpha'+4a\alpha'\frac{d\alpha'}{dt}+\alpha'^3 = 0.$$

In the small terms we may take $\alpha' = -g\int a\,dt = -\dfrac{gA}{n}\sin nt$, so that

$$\frac{d^2\alpha'}{dt^2}+g\alpha'+\frac{g^2A^3}{4n}(\sin nt+5\sin 3nt) = 0.$$

To satisfy this we assume

$$\alpha' = H\sin nt+K\sin 3nt.$$

Then

$$H(g-n^2)+\frac{g^2A^3}{4n} = 0, \qquad K(g-9n^2)+\frac{5g^2A^3}{4n} = 0,$$

from the first of which

$$n^2 = g+\frac{g^2A^3}{4nH} = g-\frac{gA^2}{4}, \tag{41}$$

or, if we restore homogeneity by introduction of k,

$$n^2 = g/k\,.\,(1-\tfrac{1}{4}k^2A^2). \tag{42}$$

With this value of n the stationary vibration

$$y = \mathrm{A} \cos nt \cos kx - \tfrac{1}{2} k\mathrm{A}^2 \cos^2 nt \cos 2kx + \tfrac{3}{8} k^2\mathrm{A}^3 \cos^3 nt \cos 3kx, \qquad (43)$$

satisfies all the conditions. It may be remarked that according to (42) the frequency of vibration is diminished by increase of amplitude.

The special cases above considered of purely progressive or purely stationary waves possess an exceptional simplicity. In general, with omission of β, β', equations (28), (29), become

$$\frac{d^2\alpha}{dt^2} + g\alpha - \frac{2}{g}\frac{d\alpha}{dt}\frac{d(\alpha^2+\alpha'^2)}{dt} + \alpha(\alpha^2+\alpha'^2) = 0, \qquad (44)$$

and a like equation in which α and α' are interchanged. In the terms of the third order, we take

$$\alpha = \mathrm{P} \cos nt + \mathrm{Q} \sin nt, \qquad \alpha' = \mathrm{P}' \cos nt + \mathrm{Q}' \sin nt, \qquad (45)$$

so that

$$\alpha^2+\alpha'^2 = \tfrac{1}{2}(\mathrm{P}^2+\mathrm{Q}^2+\mathrm{P}'^2+\mathrm{Q}'^2) + \tfrac{1}{2}(\mathrm{P}^2+\mathrm{P}'^2-\mathrm{Q}^2-\mathrm{Q}'^2)\cos 2nt$$
$$+ (\mathrm{PQ}+\mathrm{P}'\mathrm{Q}') \sin 2nt.$$

The third order terms in (44) are

$$\begin{aligned}
&\tfrac{1}{2}(\mathrm{P}^2+\mathrm{P}'^2+\mathrm{Q}^2+\mathrm{Q}'^2)(\mathrm{P}\cos nt + \mathrm{Q}\sin nt) \\
&\quad + 2\cos nt \cos 2nt \left\{\tfrac{1}{4}\mathrm{P}(\mathrm{P}^2+\mathrm{P}'^2-\mathrm{Q}^2-\mathrm{Q}'^2) - \frac{2n^2\mathrm{Q}}{g}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}')\right\} \\
&\quad + 2\sin nt \sin 2nt \left\{\tfrac{1}{2}\mathrm{Q}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}') - \frac{n^2\mathrm{P}}{g}(\mathrm{P}^2+\mathrm{P}'^2-\mathrm{Q}^2-\mathrm{Q}'^2)\right\} \\
&\quad + 2\sin nt \cos 2nt \left\{\tfrac{1}{4}\mathrm{Q}(\mathrm{P}^2+\mathrm{P}'^2-\mathrm{Q}^2-\mathrm{Q}'^2) + \frac{2n^2\mathrm{P}}{g}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}')\right\} \\
&\quad + 2\cos nt \sin 2nt \left\{\tfrac{1}{2}\mathrm{P}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}') + \frac{n^2\mathrm{Q}}{g}(\mathrm{P}^2+\mathrm{P}'^2-\mathrm{Q}^2-\mathrm{Q}'^2)\right\},
\end{aligned}$$

of which the part in $\sin nt$ has the coefficient

$$\mathrm{Q}\{\tfrac{1}{4}(\mathrm{P}^2+\mathrm{P}'^2) + \tfrac{3}{4}(\mathrm{Q}^2+\mathrm{Q}'^2)\} + \tfrac{1}{2}\mathrm{P}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}')$$
$$+ n^2/g \,.\, \{\mathrm{Q}(\mathrm{P}^2+\mathrm{P}'^2-\mathrm{Q}^2-\mathrm{Q}'^2) - 2\mathrm{P}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}')\},$$

or, since $n^2 = g$ approximately,

$$\mathrm{Q}\{\tfrac{5}{4}(\mathrm{P}^2+\mathrm{P}'^2) - \tfrac{1}{4}(\mathrm{Q}^2+\mathrm{Q}'^2)\} - \tfrac{3}{2}\mathrm{P}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}'). \qquad (46)$$

In like manner the coefficient of $\cos nt$ is

$$\mathrm{P}\{\tfrac{5}{4}(\mathrm{Q}^2+\mathrm{Q}'^2) - \tfrac{1}{4}(\mathrm{P}^2+\mathrm{P}'^2)\} - \tfrac{3}{2}\mathrm{Q}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}'), \qquad (47)$$

differing merely by the interchange of P and Q.

But when these values are employed in (44), it is not, in general, possible, with constant values of P, Q, P′, Q′, to annul the terms in $\sin nt$, $\cos nt$. We obtain from the first

$$n^2 = g + \tfrac{5}{4}(\mathrm{P}^2+\mathrm{P}'^2) - \tfrac{1}{4}(\mathrm{Q}^2+\mathrm{Q}'^2) - \frac{3\mathrm{P}}{2\mathrm{Q}}(\mathrm{PQ}+\mathrm{P}'\mathrm{Q}'), \qquad (48)$$

and from the second

$$n^2 = g + \tfrac{5}{4}(\mathrm{Q}^2 + \mathrm{Q}'^2) - \tfrac{1}{4}(\mathrm{P}^2 + \mathrm{P}'^2) - \frac{3\mathrm{Q}}{2\mathrm{P}}(\mathrm{PQ} + \mathrm{P}'\mathrm{Q}'); \tag{49}$$

and these are inconsistent, unless

$$(\mathrm{PP}' + \mathrm{QQ}')(\mathrm{PQ}' - \mathrm{P}'\mathrm{Q}) = 0. \tag{50}$$

The latter condition is unaltered by interchange of dashed and undashed letters, and thus it serves equally for the equation in α'.

The two alternatives indicated in (50) correspond to the particular cases already considered. In the first ($\mathrm{PP}' + \mathrm{QQ}' = 0$) we have a purely progressive wave and in the second a purely stationary one.

When the condition (50) does not hold good, it is impossible to satisfy our equations as before with constant values of n, P, Q, P′, Q′; and it is perhaps hardly worth while to pursue the more complicated questions which then arise. It may suffice to remark that an approximately stationary wave can never pass into an approximately progressive wave, nor *vice versâ.* The progressive wave has momentum, while the stationary wave has none, and momentum is necessarily conserved.

When β, β', γ, γ', are not zero, additional terms enter. Equations (26), (30), show that the additions to b, b', vary as the sine and cosine of $\sqrt{(2g)}\,.\,t$, and represent waves which might exist in the complete absence of the principal wave.

The additions to c, c', are more complicated. As regards the parts depending in (31), (32), on $d\gamma/dt$, $d\gamma'/dt$, they are proportional to the sine and cosine of $\sqrt{(3g)}\,.\,t$, and represent waves which might exist alone. But besides these there are other parts, analogous to the combination-tones of Acoustics, resulting from the interaction of the β-waves with the principal wave. These vary as the sine and cosine of $\sqrt{g}\,.\,\{\sqrt{2} \pm 1\}\,t$, thus possessing frequencies differing from the former frequencies. Similar terms will enter into the expression for n^2 as determined from (28), (29).

In the particular case of β, β', vanishing, even though γ, γ' (assumed still to be of the third order), remain, we recover most of the former simplicity, the only difference being the occurrence in c, c', of terms in $\sqrt{(3g)}\,.\,t$, such as might exist alone.